ON THE CONJUGACY PROBLEM FOR CYCLIC EXTENSIONS OF FREE GROUPS

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Abstract. We study the conjugacy problem in cyclic extensions of free groups. It is shown that the conjugacy problem is solvable in split extensions of finitely generated free groups by virtually inner automorphisms. An algorithm for construction of the unique representative (the conjugacy normal form) for each conjugacy class is given.

1. Introduction

One of fundamental problems in the combinatorial group theory is the conjugacy problem formulated by M. Dehn (1912) together with two other problems: the word problem and the isomorphism problem [33, Ch. 2, § 1]. M. Dehn proved that the conjugacy problem is solvable in fundamental groups of closed orientable surfaces. Later his method have been extended to the class of groups with small cancellation [33, Ch. 5].

Let us recall basic results about solvability of the conjugacy problem in some classes of groups.

In [25] M. Gromov introduced a class of groups which are now referred to as word hyperbolic groups. Among examples of word hyperbolic groups are finite groups, free groups, small cancellation groups satisfying a metric small cancellation condition $C'(\lambda)$ with $0 < \lambda \leq 1/6$. In particular, the fundamental group of an oriented surface of genus $g > 1$ is hyperbolic. It is known [13] that the conjugacy problem in word hyperbolic groups is solvable. Also it is solvable in such generalizations of word hyperbolic groups as relative hyperbolic groups and semi-hyperbolic groups. At the same time, word hyperbolic groups are contained in the class of bi-automatic groups, which are contained in the class of automatic groups itself. The class of automatic groups belong to the class of combable group. It is known that the conjugacy problem is solvable in bi-automatic groups. Recently, M.R. Bridson [12] demonstrated that there exist combable groups in which the conjugacy problem is unsolvable. The question about solvability of the conjugacy problem in automatic groups is still open.

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We will interested in groups $F_n(t) = F_n \rtimes \langle t \rangle$ which are semi-direct products of a free group F_n and a cyclic group $\langle t \rangle$, where conjugation by t induces an automorphism $\varphi \in \mathrm{Aut}(F_n)$. In particular, if t is of infinite order, then $F_n(t)$ is the mapping torus of F_n corresponding to the automorphism φ and is denoted by

$$F_n(\varphi) = \langle x_1, x_2, \ldots, x_n, t \,|\, t^{-1} x_i t = \varphi(x_i), \quad i = 1, 2, \ldots, n \rangle.$$

If t is of finite order (which is divided by order of φ) then there exists a homomorphism of $F_n(\varphi)$ onto $F_n(t)$.

The study of groups $F_n(\varphi)$ is also motivated by their relation with fundamental groups of closed 3-manifolds fibering over a circle (see [20]).

In [23] J.M. Gersten and J.R. Stallings asked about word hyperbolicity of $F_n(\varphi)$. It was shown by M. Bestvina and M. Feighn [5] and by P. Brinkmann [15] that $F_n(\varphi)$ is word hyperbolic if and only if φ having no nontrivial periodic conjugacy classes (i.e. there no non-trivial element f of F_n and non-zero integer k such that $\varphi^k(f)$ is conjugated to f in F_n). The solvability of the conjugacy problem for such groups follows from the solvability of the conjugacy problem in word hyperbolic groups.

We remark that some one-relator groups as well as some Artin groups can be presented as cyclic extensions of free groups (see discussions in Section 2). The conjugacy problem in one-relator groups in still open. At the same time, it is solvable in one-relator groups with torsion. This result was announced by B. Newman [36]. S. Pride [39] proved this fact for the case when the defining relation is of the form r^n for $n > 2$. Another proof of this fact was given by V.N. Bezverhnii [6]. L. Larsen [32] proved that the conjugacy problem is solvable in one-relator groups with non-trivial center. For some classes of one-relator groups the conjugacy problem was solved by G.A. Gurevich [26, 27] and A.A. Fridman [21].

It is known that the conjugacy problem is solvable in braid groups and knot groups [22, 40, 41]. The solvability of the conjugacy problem in link groups seems still open.

Recall that the conjugacy problem is solvable in the Novikov group A_{p_1,p_2} [38] if and only if the word problem is solvable in the corresponding Post system $P(A_{p_1,p_2})$ [9], [10, Ch. 7].

Surveys on the conjugacy problem in various classes of groups can be found in [28] and [37]. The problem on solvability of the conjugacy problem in groups $F_n(\varphi)$ was formulated also by I. Kapovich [29, Problem 6.2].

It is known that the solvability of the conjugacy problem does not preserve under finite extensions [24]. Moreover, D. Collins and C. Miller [16] constructed a group G containing a subgroup H of index two such that the conjugacy problem is solvable in H but not solvable in G. At the same paper they constructed a group with solvable conjugacy problem which contains a subgroup of index two with non-solvable conjugacy problem.

One of possible approaches to solve the conjugacy problem in groups $F_n(t)$ is investigation of the property to be conjugacy separable.

A group G is said to be *conjugacy separable* if for any pair of non-conjugated elements of G there exists a homomorphism of G to a finite group such that images of these elements are also non-conjugated. It was shown by A.I. Malcev [34], if a finitely generated group is conjugacy separable then the conjugacy problem in this group is solvable. Thus, the following problem aries naturally:

Problem. *Is a group $F_n(t)$ conjugacy separable?*

It was shown by J.L. Dyer [18] that finite extensions of free groups are conjugacy separable. Also, he proved in [19] that if G is an extension of a free group by a cyclic group and center of G is nontrivial, then G is conjugacy separable. Moreover, if G is one-relator group with non-trivial center, then G is conjugacy separable.

In the present paper we consider groups $G = F_n \leftthreetimes \langle t \rangle$ which are semi-direct product of a free group F_n and a cyclic group $\langle t \rangle$ such that the conjugation by t induces automorphism $\varphi \in \mathrm{Aut}(F_n)$ such that $\varphi^m \in \mathrm{Inn}(F_n)$, i. e. φ^m is inner automorphism of F_n for some positive integer m. Such automorphism φ will be referred to as *virtually inner*. In Section 2 we will demonstrate that many one-relator groups and, in particular, two-generator Artin groups can be obtained as split extension of free groups by virtually inner automorphisms (see Section 2). In Section 3 and Section 4 we will show that the conjugacy problem is solvable in G. For each element of G we will construct unique conjugacy normal from. If t is of finite order, then G is word hyperbolic, and moreover, in virtue of the above referred result of J.L. Dyer, it is conjugacy separable. Therefore, the conjugacy problem is solvable in G. But our approach gives the more effective solving algorithm than the general solving algorithm for word hyperbolic groups. If t is of infinite order, then G is not word hyperbolic because it contains a subgroup isomorphic to $\mathbb{Z} \times \mathbb{Z}$ (see [19]). In Section 5 we will show that the conjugacy problem is solvable for the split extension $F_\infty \ltimes \mathbb{Z}$ of the countably generated free group F_∞ by the automorphism φ shifting its generators.

2. One-relator groups and 2-generated Artin groups

In this section we will show that some one-relator groups and, in particular, 2-generator Artin groups can be obtained as split extensions of free groups by virtually inner automorphisms.

It was shown in [2, 7, 8] that the conjugacy problem is solvable in Artin and Coxeter groups of large type. Also, it was shown in [14] that it is solvable for Artin groups of finite type, i.e. such that corresponding Coxeter groups are finite. The question about solvability of the conjugacy problem in arbitrary Artin group seems still open.

Consider a group

$$G = \langle t, a \mid r = 1 \rangle$$

with two generators and one defining relation. Let us assume that the word r is cyclically reduced and contains both generators a and t.

Applying, if necessary, Lemma 11.8 from [33, ch. 5] we can assume that the exponent sum of t in r is equal to zero. We will use the Magnus–Moldavanskii method to represent G as HNN-extension. Let us define new generators $a_i = t^{-i}at^i$, $i \in \mathbb{Z}$ and let r' be the presentation of the word r written in these generators. We write $a_i \in r'$ if a_i or a_i^{-1} is a subword of r', and denote $\mu = \min\{i \mid a_i \in r'\}$ and $\nu = \max\{i \mid a_i \in r'\}$. Let us define

$$\begin{aligned} H &= \langle a_i,\, \mu \le i \le \nu \mid r' = 1\rangle, \\ A &= \langle a_i,\, \mu \le i < \nu\rangle, \\ B &= \langle a_i,\, \mu < i \le \nu\rangle. \end{aligned}$$

Then we can write $G = \langle H, t \mid t^{-1}At = B, \varphi\rangle$, where isomorphism φ acts as the following

$$\varphi(a_i) = a_{i+1},\ i = \mu, \mu+1, \ldots, \nu-1.$$

If a_ν appears in r' only once (with degree $+1$ or -1), then using r' we can express a_ν via other generators and eliminate it from the generating system for H. Thus H is a free generated group which coincides with A. In this case

$$G = \langle H, t \mid t^{-1}Ht = B, \varphi\rangle$$

is said to be an *ascending HNN-extension* of H. If, moreover, $B = H$, then φ is an automorphism of H, and $G = F_n(\varphi)$, where $n = \nu - \mu - 1$, is a cyclic extension of a free group.

Analogously, if a_μ can be expressed from r' via other generators, we get that H coincides with B.

As a noticeable example, let us consider 2-generated Artin group:

$$\mathcal{A}(m) = \langle x, y \mid w_m(x,y) = w_m(y,x)\rangle,$$

where $m \ge 3$ is integer and

$$w_m(u,v) = \begin{cases} (uv)^n, & \text{if} \quad m = 2n, \\ (uv)^n\, u, & \text{if} \quad m = 2n+1. \end{cases}$$

We recall that $\mathcal{A}(2n+1)$ is the fundamental group of the $(2n+1,2)$-torus knot complement in the 3-sphere and $\mathcal{A}(2n)$ is the fundamental group of the $(2n,2)$-torus 2-component link complement in the 3-sphere. These knots and links arise as closures of 2-strand braids.

Changing generators of $\mathcal{A}(m)$ in the same way as in [11] (where the Gröbner–Shirshov bases for these groups were constructed) we will get the following result.

Proposition 2.1. *Any 2-generated Artin group is a split extension of a free group of finite rank and a cyclic group generated by a virtually inner automorphism.*

Proof. Let us consider the group $\mathcal{A}(2n) = \langle x, y \mid (xy)^n = (yx)^n\rangle$, $n \ge 2$, Denoting $t = x$ and $y_i = t^iyt^{-i}$, where $i = 0, 1, \ldots, n-1$, we get the following presentation:

$$\mathcal{A}(2n) = F_n(\varphi) = \langle y_0, y_1, \ldots, y_{n-1}, t \mid t^{-1}y_it = \varphi(y_i)\,, \quad i = 0, \ldots, n-1\rangle,$$

where F_n is the free group generated by $y_0, \ldots, y_{n-1}$ and φ is defined by:

$$\begin{aligned} \varphi(y_0) &= y_0 y_1 \cdots y_{n-2} y_{n-1} y_{n-2}^{-1} \cdots y_1^{-1} y_0^{-1}, \\ \varphi(y_i) &= y_{i-1}, \qquad\qquad i = 1, 2, \ldots, n-1. \end{aligned}$$

It is easy to check that the element $\Delta = y_0 y_1 \cdots y_{n-1}$ is such that $\varphi(\Delta) = \Delta$ (see also [11]) and the automorpihsm φ^n acts as the following:

$$\varphi^n(y_i) \,=\, \Delta\, y_i\, \Delta^{-1}, \qquad i = 0, 1, \ldots, n-1,$$

so φ is an virtually inner automorphism of F_n.

Let us consider the group $\mathcal{A}(2n+1) = \langle x, y \mid (xy)^n x = (yx)^n y \rangle$. Denoting $t = x$, $z = yx^{-1} = yt^{-1}$, and $z_i = t^i z t^{-i}$, where $i = 0, 1, \ldots, 2n-1$, we get the following presentation

$$\mathcal{A}(2n+1) = F_{2n}(\psi) = \langle z_0, \ldots, z_{2n-1}, t \mid t^{-1} z_i t = \psi(z_i), \quad i = 0, \ldots, 2n-1 \rangle,$$

where F_{2n} is the free group generated by $z_0, z_1, \ldots, z_{2n-1}$ and ψ is defined by:

$$\begin{aligned} \psi(z_0) &= z_0 z_2 \cdots z_{2n-2} z_{2n-1}^{-1} z_{2n-3}^{-1} \cdots z_3^{-1} z_1^{-1}, \\ \psi(z_i) &= z_{i-1}, \qquad\qquad i = 1, \ldots, 2n-1. \end{aligned}$$

It is easy to check that the element

$$\Sigma \,=\, z_0 z_2 \cdots z_{2n-2} \,(z_0 z_1 \cdots z_{2n-1})^{-1}\, z_1 z_3 \cdots z_{2n-1}$$

is such that $\psi(\Sigma) = \Sigma$ (see also [11]) and the automorphism $\psi^{2(2n+1)}$ acts as the following:

$$\psi^{2(2n+1)}(z_i) \,=\, \Sigma\, z_i\, \Sigma^{-1}, \qquad i = 0, 1, \ldots, 2n-1,$$

so, ψ is an virtually inner automorphism of F_n. □

We remark that the conjugacy problem in 2-generator Artin groups is solvable since these groups are Artin groups of finite type.

3. Finitely generated free groups and virtually inner automorphisms

Let $F_n = \langle x_1, x_2, \ldots, x_n \rangle$ be the free group of rank $n \geq 2$ with words from the alphabet $\mathbb{X} = \{x_1^{\pm 1}, \ldots, x_n^{\pm 1}\}$. In some cases, we will need to distinguish words which represent the same element of the group. We will write $U = V$ if two words (or two elements of the group) are equal as elements of the group, and $U \equiv V$ if two words are equal graphically. Denote by $|V|$ the length of a word V in the alphabet $\mathbb{X}$. A word V is said to be *reduced* if it contains no part xx^{-1}, $x \in \mathbb{X}$. A reduced word V defines a non-identity element if and only if $|V| \geq 1$. A reduced word obtained by reducing of an original word will be referred to as its *reduction*. By $||V||$ we will denote the length of the reduction of a word V. Further, a reduced word $V = x_{i_1}^{\varepsilon_1} x_{i_2}^{\varepsilon_2} \cdots x_{i_n}^{\varepsilon_n}$, where $\varepsilon_i = \pm 1$, $i = 1, \ldots, n$, is said to be *cyclically reduced* if $i_1 \neq i_n$ or if $i_1 = i_n$ then $\varepsilon_1 \neq -\varepsilon_n$. Clearly, every element of a free group is conjugated

to an element given by a cyclically reduced word referred to as its *cyclic reduction.*

We will use standard notations $\mathrm{Aut}(F_n)$ and $\mathrm{Inn}(F_n)$ for the group of automorphisms and the group of inner automorphisms of F_n, respectively. An automorphism $\varphi \in \mathrm{Aut}(F_n)$ is said to be *virtually inner* if $\varphi^m \in \mathrm{Inn}(F_n)$ for some positive integer m.

In particular, any automorphism φ of finite order is virtually inner.

Denote by $F_n(t) = F_n \ltimes \langle t \rangle$ the semi-direct product, where the generator t of the cyclic group $\langle t \rangle$ is such that the conjugation by t induces an automorphism φ, i.e. $t^{-1} f t = \varphi(f)$ for any $f \in F_n$. Remark that $F_n \triangleleft F_n(t)$.

We will show that the following property holds.

Theorem 3.1. *If φ is a virtually inner automorphism of a free group F_n, $n \geq 2$, then the conjugacy problem is solvable for $F_n(t)$.*

To prove this result, we will find an unique representative for each conjugacy class.

Any element $v \in F_n(t)$ can be presented in the form $t^{\ell} V$ where $\ell \in \mathbb{Z}$ if t has infinite order, and $0 \leq \ell < |t|$ if t has finite order $|t|$; V is a word of the alphabet $\mathbb{X}$ (we will also say that V is a $\mathbb{X}$-*part* of v). Moreover, if V is reduced then such a presentation of v is unique.

Let $\varphi \in \mathrm{Aut}(F_n)$ be a virtually inner automorphism such that $\psi = \varphi^m$ is an inner automorphism of F_n. Without loss of generality we can assume that m is taken the smallest positive integer having such a property. There exists a reduced word $\Delta \in F_n$ such that

$$\varphi^m(f) \; = \; \Delta^{-1} f \Delta$$

for any $f \in F_n$. It is easy to check (see also [3, Lemma 2]) that following properties hold.

Lemma 3.1. *(1) Automorphisms φ and ψ commute.*
(2) If $k = mq + r$, where $q \in \mathbb{Z}$ and $0 \leq r \leq m - 1$, then for any word U of the alphabet $\mathbb{X}$ we have

$$\varphi^k(U) \; = \; \varphi^r(\Delta^{-q} \, U \, \Delta^q).$$

(3) $\varphi(\Delta) = \Delta$.

Let us define a linear order "<" on the set of irreducible words in the alphabet $\mathbb{X}$. Assume that elements of $\mathbb{X}$ are ordered in the following way:

$$x_1 \, < \, x_1^{-1} \, < \, x_2 \, < \, x_2^{-1} \, < \, \ldots \, < \, x_n \, < \, x_n^{-1}.$$

We write $U < V$ if $|U| < |V|$ or if $|U| = |V|$ and the word U is less than the word V in respect to the lexicographical order corresponding to the above defined linear order on $\mathbb{X}$.

A reduced word $V \in F_n$ is said to be Δ-*reduced* if $|V| \leq ||\Delta^{-k} V \Delta^k||$ for all $k \in \mathbb{Z}$. Obviously, if V is cyclically reduced, then the length of any word conjugated to V is not less than the length of V, so V is Δ-reduced.

Lemma 3.2. *Suppose that Δ is cyclically reduced. A reduced word $V \in F_n$ is Δ-reduced if $|V| \le ||\Delta^{-\varepsilon} V \Delta^{\varepsilon}||$ for $\varepsilon = \pm 1$.*

Proof. See [3, Lemma 3]. □

If Δ is cyclically reduced, Lemma 3.2 gives the finite algorithm to find for a given reduced word V a Δ-reduced word V_Δ conjugated to V by some power of Δ. Indeed, it is enough to repeat conjugations of V by Δ^ε, $\varepsilon = \pm 1$, few times. If the length of the obtained word is less than the length of the previous word, we will conjugate again. If not, then the obtained word is a Δ-reduced word V_Δ conjugated to V. Such a construction of a Δ-reduced word V_Δ conjugated to V by some power of Δ will be referred to as a Δ-*reduction.*

We remark that if Δ is not cyclically reduced, then the analog of Lemma 3.2 does not hold. It is clear from the following example.

Example. Let $U, W, \Sigma \in F_n$ be nonempty reduced words such that for an integer $|k| > 1$ words $\Delta \equiv U^{-1}W^{-1}\Sigma W U$ and $V \equiv U^{-1}W^{-1}\Sigma^k W U^2$ are reduced. If $k > 1$, we get

$$\begin{array}{rcl} \Delta^{-1} V \Delta & = & U^{-1}W^{-1}\Sigma^{k-1}WUW^{-1}\Sigma WU, \\ \Delta^{-2} V \Delta^2 & = & U^{-1}W^{-1}\Sigma^{k-2}WUW^{-1}\Sigma^2 WU, \\ & \dots & \\ \Delta^{-(k-1)} V \Delta^{k-1} & = & U^{-1}W^{-1}\Sigma WUW^{-1}\Sigma^{k-1} WU, \\ \Delta^{-k} V \Delta^k & = & W^{-1}\Sigma^k WU. \end{array}$$

It is easy to see that

$$|V| < ||\Delta^{-1} V \Delta|| = ||\Delta^{-2} V \Delta^2|| = \ldots = ||\Delta^{-(k-1)} V \Delta^{k-1}||,$$

but $||\Delta^{-k} V \Delta^k|| < |V|$.

If $k < -1$, similar example can be obtained.

Lemma 3.3. *Suppose Δ is not cyclically reduced. Let V be a reduced word such that $|V| \le ||\Delta^{-\varepsilon} V \Delta^{\varepsilon}||$, $\varepsilon = \pm 1$. Then one of the following cases holds:*
(1) V is Δ-reduced.
(2) $\Delta \equiv U_1^{-1}U_2^{-1}\Delta_{11}^{-1}\Delta_2\Delta_{11}U_2U_1$ and $V \equiv U_1^{-1}U_2^{-1}WU_2U_1$, for some reduced $\Delta_{11}, \Delta_2, U_1, U_2$ and cyclically reduced W for which there exist integers k, $|k| > 1$, and $m \ge 0$ and reduced Φ, W_0 such that either
a) $U_2^{-1}\Delta_{11}^{-1}\Delta_2^{-k}\Delta_{11} \equiv \Phi W^{-m}$ and $W \equiv \Phi^{-1}W_0$,
or
b) $\Delta_{11}^{-1}\Delta_2^{k}\Delta_{11}U_2 \equiv W^{-m}\Phi$ and $W \equiv W_0\Phi^{-1}$.
In addition, Φ does not end by W^{-1}, and W_0 is either nonempty or $W_0 \equiv 1$ with $U_1^{-1}\Phi^{-1}U_1$ be reduced.

In case (2) $V_\Delta = \Delta^{-k} V \Delta^k$ is Δ-reduced word conjugated to V. Moreover, case (2) describe all possible cases when $|V| \le ||\Delta^{-\varepsilon} V \Delta^{\varepsilon}||$, $\varepsilon = \pm 1$, but $||\Delta^{-k} V \Delta^k|| < |V|$ for some $|k| > 1$.

Proof. See [3, Lemma 4]. □

If Δ is not cyclically reduced, Lemma 3.3 gives the finite algorithm to find for a given reduced word V a Δ-reduced word V_Δ conjugated to V by some power of Δ. If Δ and V are of the form represented in case (2) of Lemma 3.3, then we define $V_\Delta = \Delta^{-k} V \Delta^k$. In this case we say that V_Δ is a Δ-*reduction* of V. If Δ and V are others, then we follow the same steps as described after Lemma 3.2.

Remark that V_Δ is not uniquely determined by V (see [3, Proposition 2]).

4. Constructing of conjugated normal form

Now we will study Δ-reduced elements of the group $F_n(t)$. A word $v = t^\ell V \in F_n(t)$, $\ell \in \mathbb{Z}$, $V \in F_n$, is said to be Δ-*reduced* if V is Δ-reduced.

In virtue of Lemma 3.1(3), the conjugation of $v = t^\ell V$ by any power Δ^m means the conjugation of V by Δ^m, i.e.

$$\begin{array}{rcccl} \Delta^{-m} v \Delta^m & = & \Delta^{-m} t^\ell V \Delta^m & = & t^\ell (t^{-\ell} \Delta^{-m} t^\ell) V \Delta^m \\ & = & t^\ell \varphi^\ell (\Delta^{-m}) V \Delta^m & = & t^\ell \Delta^{-m} V \Delta^m. \end{array}$$

For any integer k we will denote by $V_{[k]}$ a reduction of $t^{-k} V t^k = \varphi^k(V)$. Remark that $t^{-m} V_{[l]} t^m = V_{[l+m]}$ and that $V_{[0]} = V$ if V is a reduced word.

If length of a reduced word V in the alphabet $\mathbb{X}$ is bigger than 1 then it can be presented as a product $V \equiv V' V''$ of two nonempty reduced words. The word V' will be referred to as an *initial part* of V and V'' will be referred as a *final part* of V. Denote by $\mathcal{I}(V)$ the set of all initial parts of V and by $\mathcal{F}(V)$ the set of all final parts of V.

Consider $v \in F_n(t)$ and fix its presentation $v \equiv t^\ell V$, $\ell \in \mathbb{Z}$, where V is reduced word in the alphabet $\mathbb{X}$. Conjugation by t^ℓ induces an automorphism $\psi = \varphi^\ell$ of the free group F_n.

For a reduced word $V \equiv V' V''$ in the alphabet $\mathbb{X}$ with $V' \in \mathcal{I}(V)$, $V'' \in \mathcal{F}(V)$ we say that a word $V''_{[\ell]} V'$ is a *cyclic ψ-shift of a final part of* V and a word $V'' V'_{[-\ell]}$ is a *cyclic ψ-shift of an initial part of* V. Remark that these elements can be obtained by a conjugation of v. Indeed, conjugating v by $(V'')^{-1}$ we will get

$$V'' v (V'')^{-1} = t^\ell t^{-\ell} V'' t^\ell V' = t^\ell V''_{[\ell]} V',$$

and conjugating v by $(V')_{[-\ell]}$ we will get

$$(V'_{[-\ell]})^{-1} v V'_{[\ell]} = t^\ell t^{-\ell} (V'_{[-\ell]})^{-1} t^\ell V' V'' V'_{[-\ell]} = t^\ell V'' V'_{[-\ell]}.$$

If $|V'| = 1$, i.e. $V' = x_i^\varepsilon$, $\varepsilon = \pm 1$, then the corresponding cyclic ψ-shift will be referred to as a *cyclic ψ-shift of the initial letter*. If $|V''| = 1$ then the corresponding cyclic ψ-shift will be referred to as a *cyclic ψ-shift of the final letter*.

A reduced word V will be referred to as a *cyclically ψ-reduced* if there doesn't exist a cyclic ψ-shift of its final part and there doesn't exist a cyclic ψ-shift of its initial part which decreases length of V. Obviously, applying

cyclic ψ-shifts of final (or initial) parts to a given word V we will get a cyclically ψ-reduced word conjugated to V.

Now let us construct conjugating normal form for an element $v = t^\ell V$. Without loss of generality, we can assume that V is cyclically ψ-reduced and Δ-reduced. For a given V let us construct a set of words $\mathcal{V}_\Delta$ which contains V and such elements which are conjugated to V by elements of the group $\langle\Delta\rangle$ and are Δ-reduced. All of them have the same length as V. By Lemma 3.2 and Lemma 3.3 the set $\mathcal{V}_\Delta$ is finite. Applying to elements of $\mathcal{V}_\Delta$ cyclic ψ-shifts of all initial parts and all final parts, we will construct the set $(\mathcal{V}_\Delta)_\psi$. Applying to obtained words Δ-reducing, we will get a set $((\mathcal{V}_\Delta)_\psi)_\Delta$ each element of which is Δ-reduced. Consider the subset of $((\mathcal{V}_\Delta)_\psi)_\Delta$ consisting of cyclically ψ-reduced words. Multiplying each of obtained words on left by t^ℓ we will get a set of words from $F_n(t)$. Let us denote the obtained set by $D_0(v)$.

Remark that in the free group the set of words obtained from a given word V by cyclic shifts is finite. But the set of words obtained from a word $v \in F_n(t)$ by cyclic ψ-shifts can be infinite if t has infinite order. Indeed, it is clear from following relations

$$\psi^{k-1}(V)\,\psi^{k-2}(V)\,\cdots\,\psi(V)\,V\,v\,V^{-1}\,\psi(V^{-1})\,\cdots\,\psi^{k-2}(V^{-1})\,\psi^{k-1}(V^{-1}) = t^\ell\,\psi^k(V), \qquad \text{for} \qquad k > 0;$$

$$\psi^{k}(V^{-1})\,\psi^{k+1}(V^{-1})\,\cdots\,\psi^{-1}(V^{-1})\,v\,\psi^{-1}(V)\,\cdots\,\psi^{k+1}(V)\,\psi^{k}(V) = t^\ell\,\psi^k(V), \qquad \text{for} \qquad k < 0,$$

that the element $v \equiv t^\ell V$ is conjugated to $t^\ell\psi^k(V)$ for any integer k. Moreover, the conjugation can be done by an element of the free group F_n.

For each integer k we define a set $D_k(v) = D_0(t^\ell\psi^k(V))$, and $D(v) = \cup_{k\in\mathbb{Z}} D_k(v)$. Let us verify that $D(v)$ is finite.

Lemma 4.1. *For integer ℓ and m as above denote $d = \gcd(m, \ell)$. Then*

$$D(v) \,=\, \cup_{k\in\{0,d,\dots,m-d\}} D_0(t^\ell\varphi^k(V)).$$

Proof. For any integer k we have $\psi^k(V) = \varphi^{\ell k}(V)$. Let r, $0 \le r < m$ be such that $\ell k = mq + r$, $q \in \mathbb{Z}$. By Lemma 3.1,

$$\varphi^{\ell k}(V) \,=\, \Delta^{-q}\,\varphi^r(V)\,\Delta^q.$$

Let $\ell = \ell_1 d$ and $m = m_1 d$ for some integer ℓ_1 and m_1. If k runs over the set $\mathbb{Z}$ then r runs over the set $\{0, d, 2d, \dots, m-d\}$. Let us show that $D_k(v) = D_0(t^\ell\varphi^r(V))$, that will give the statement. Indeed, by the definition, $D_k(v) = D_0(t^\ell\Delta^{-q}\varphi^r(V)\Delta^q)$. Denote $U = V_{[r]} = \varphi^r(V)$ and consider elements from $D_0(t^\ell\varphi^r(V))$. By the definition, $D_0(t^\ell\varphi^r(V))$ consists of words whose $\mathbb{X}$-parts are the ψ-shifts $\psi(U_2)U_1$ or $U_2\psi^{-1}(U_1)$, where $U \equiv U_1U_2$, to which Δ-reducing and ψ-reducing are applied. To construct $D_k(v)$ we must pass from a word $\Delta^{-q}\,\varphi^r(V)\,\Delta^q$ to a Δ-reduced word, which, in a general case, can be different from $U = \varphi^r(V)$. But, according to the definition of

$D_0(v)$, the set of Δ-reduced words constructed from $\Delta^{-q}\,\varphi^r(V)\,\Delta^q$ coincides with the set of Δ-reduced words constructed from $\varphi^r(V)$. So, corresponding sets of all cyclic ψ-shifts of initial parts and of finals parts also coincide. Therefore, $D_k(v) = D_0(t^\ell \varphi^r(V))$. □

Lemma 4.2. *The set $D(v)$ has the following properties:*
(1) If $u \in D(v)$ then $D(u) = D(v)$;
(2) Let elements $v \equiv t^\ell V$ and $w \equiv t^\ell W$ be conjugated by an element of the group $H = \langle F_n, t^m\rangle$. Suppose that words V and W are cyclically ψ-reduced and Δ-reduced. Then $D(v) = D(w)$.

Proof. Since $D(v)$ consists of elements conjugated by elements of H, item (2), obviously, implies (1). Let us prove (2). Let $u \equiv t^{ms}U$ be a conjugating element, where s is some integer and U is a reduced word. By the assumption, $w = u^{-1}vu$, i.e. we have the following equality in the free group F_n:

$$W \;=\; U_{[\ell]}^{-1}\,\Delta^{-s}\,V\,\Delta^s\,U. \tag{1}$$

Since $t^\ell\Delta^{-s} = \Delta^{-s}t^\ell$, denoting $U' = \Delta^s U$ we get $U_{[\ell]}^{-1}\Delta^{-s} = (U'_{[\ell]})^{-1}$, i.e. $W = U_{[\ell]}^{-1}\Delta^{-s}V\Delta^s U = (U'_{[\ell]})^{-1}VU'$. Therefore, without loss of generality, we can assume that in (1) the exponent s is equal to zero. Since W is cyclically ψ-reduced, the product $U_{[\ell]}^{-1}VU$ contains cancellations. Moreover, these cancellations are either in the product $U_{[\ell]}^{-1}V$ or in the product VU, but not in the both.

Case 1. Suppose that V is not cancelling wholly.

Case 1(a). Suppose that there are cancellations in the product $U_{[\ell]}^{-1}V$. Then $U_{[\ell]}^{-1}$ must be cancelled wholly, i.e. $V \equiv U_{[\ell]}V_1$. Then

$$W \;=\; U_{[\ell]}^{-1}VU \;\equiv\; U_{[\ell]}^{-1}(U_{[\ell]}V_1)U \;=\; V_1U$$

and $w \equiv t^\ell V_1 U$ is obtained from $v \equiv t^\ell U_{[\ell]}V_1$ by the shift of the initial part $U_{[\ell]}$. Indeed, according to the above described procedure, we need to conjugate v by $U = U_{[0]}$:

$$U_{[0]}^{-1}vU_{[0]} \;=\; U_{[0]}^{-1}(t^\ell U_{[\ell]}V_1)U_{[0]} \;=\; t^\ell U_{[\ell]}^{-1}U_{[\ell]}V_1U_{[0]} \;=\; t^\ell V_1 U = w.$$

Therefore, $D(v) = D(w)$.

Case 1(b). Suppose that there are cancellations in the product VU. Then U must be cancelled wholly, i.e. $V = V_1U^{-1}$. Then

$$W \;=\; U_{[\ell]}^{-1}VU \;\equiv\; U_{[\ell]}^{-1}(V_1U^{-1})U \;=\; U_{[\ell]}^{-1}V_1$$

and $w \equiv t^\ell U_{[\ell]}^{-1}V_1$ is obtained from $v \equiv t^\ell V_1U^{-1}$ by the shift of the final part U^{-1}. Indeed, conjugating v by U we get

$$U^{-1}vU \;\equiv\; U^{-1}(t^\ell V_1U^{-1})U \;=\; U_{[\ell]}^{-1}V_1.$$

Therefore, $D(v) = D(w)$.

Case 2. Suppose that V is cancelling wholly.

Case 2(a). Suppose that V is cancelling wholly in the product VU and after that there are cancellations of letters of the remaining part of U with letters of $U^{-1}_{[\ell]}$. Then we can represent $U \equiv V^{-1}U_1$, therefore $U^{-1}_{[\ell]} = U^{-1}_{1\,[\ell]}V_{[\ell]}$ and

$$U^{-1}_{[\ell]}VU \;=\; U^{-1}_{1\,[\ell]}V_{[\ell]}VV^{-1}U_1 \;=\; U^{-1}_{1\,[\ell]}V_{[\ell]}U_1. \tag{2}$$

If U_1 is cancelling wholly with $V_{[\ell]}$ then $V_{[\ell]} \equiv V_2U_1^{-1}$ and

$$U^{-1}_{1\,[\ell]}V_{[\ell]}U_1 \;\equiv\; U^{-1}_{1\,[\ell]}V_2U_1^{-1}U_1 \;=\; U^{-1}_{1\,[\ell]}V_2 \;=\; W.$$

If word $V_{[\ell]}$ in the product $V_{[\ell]}U_1$ is cancelling wholly, then we will use induction by length of U.

Remark that W arises in the process of the construction of $D_1(v)$. Indeed,

$$D_1(v) = D_0(t^\ell V_{[\ell]}) = D_0(t^\ell V_2U_1^{-1})$$

and W is obtained from $V_2U_1^{-1}$ by the cyclic ψ-shift of the final part.

Case 2(b). Suppose that V is cancelling wholly in the product $U^{-1}_{[\ell]}V$ and after that there are cancellations of letters of the remaining part of $U^{-1}_{[\ell]}$ with letters of U. Then we can represent $U^{-1}_{[\ell]} \equiv U_1^{-1}V^{-1}$, therefore $U = \psi^{-1}(V)\psi^{-1}(U_1)$ and

$$U^{-1}_{[\ell]}VU = U_1^{-1}V^{-1}VV_{[-\ell]}U_{1\,[-\ell]} \;=\; U_1^{-1}V_{[-\ell]}U_{1\,[-\ell]}. \tag{3}$$

If U_1^{-1} is cancelling wholly with $V_{[-\ell]}$ then $V_{[-\ell]} = U_1V_2$ and $U_1^{-1}V_{[-\ell]}U_{1\,[-\ell]} = U_1^{-1}(U_1V_2)U_{1\,[-\ell]} = V_2U_{1\,[-\ell]}$, i.e. $V = U_{1\,[\ell]}V_{2\,[\ell]}$ and $W \equiv V_2U_{1[-\ell]}$. Comparing these words we see that W belongs to $D_{-1}(v)$. Indeed, by the definition,

$$D_{-1}(v) \;=\; D_0(t^\ell\psi^{-1}(V)) \;=\; D_0(t^\ell U_1V_2).$$

Applying to U_1V_2 the cyclic ψ-shift of the initial part U_1, we will get $W = V_2U_{1\,[-\ell]}$. The case when $V_{[-\ell]}$ is cancelling wholly with U_1^{-1} in (3) can be considered similar to the above. □

Let us define

$$\overline{D}(v) \;=\; \cup_{0\le k<m} D(t^{-k}vt^k).$$

Obviously, this set is finite. Let $\overline{v} = t^\ell V_0$ be an element with the smallest $\mathbb{X}$-part among all elements of $\overline{D}(v)$. Such $\overline{v}$ is said to be the *conjugacy normal form* for v. By the construction, v and $\overline{v}$ are conjugated in $F_n(t)$. To show uniqueness of $\overline{v}$ we will use the following statement.

Lemma 4.3. *Let $w \equiv t^\ell W \in F_n(t)$ be cyclically ψ-reduced and Δ-reduced. If w is conjugated to v in $F_n(t)$ then $\overline{D}(w) = \overline{D}(v)$.*

Proof. Suppose that v and w are conjugated in $F_n(t)$ by $u \equiv t^kU$, where $k \in \mathbb{Z}$ and U is a reduced word in the alphabet $\mathbb{X}$. Let r, $0 \le r \le m-1$, be such that $k = mq + r$ for some integer q. By Lemma 3.1 we have

$$u^{-1}vu \;=\; U^{-1}t^{-k}(t^\ell V)t^kU \;=\; U^{-1}t^\ell(\Delta^{-q}t^{-r}Vt^r\Delta^q)U$$

$$= \quad U^{-1}t^{-mq}t^{\ell}(t^{-r}Vt^{r})t^{mq}U = U^{-1}t^{-mq}(t^{-r}vt^{r})t^{mq}U,$$

i.e. w is conjugated to $t^{-r}vt^{r}$ by an element from the group $\langle F_n, t^m\rangle$. By the construction, $D(t^{-r}vt^{r}) \subseteq \overline{D}(v)$, and by Lemma 4.2,

$$D(t^{-r}vt^{r}) = D(U^{-1}t^{-mq}(t^{-r}vt^{r})t^{mq}U).$$

Therefore, $\overline{D}(w) = \overline{D}(v)$. □

Now we are able to complete the proof of Theorem 3.1

Proof. Let $v = t^{\ell}V$ be an element of $F_n(t)$. Using, if necessary, conjugation by elements of F_n, we can assume that V is cyclically ψ-reduced and Δ-reduced. Let us construct a set $\overline{D}(v)$ as in Section 4. From this set we choose words of minimal length. Then, from such words, choose the conjugacy normal form $\overline{v}$ for v, that is the word whose $\mathbb{X}$-part is minimal in respect to the above ordering on F_n.

For a pair of given words $u = t^{k}U$ and $v = t^{\ell}V$ the conjugacy problem is solving as the following. If $k \neq \ell$ then u and v are not conjugated in $F_n(t)$. If $k = \ell$ then let us construct conjugacy normal forms $\overline{u}$ and $\overline{v}$. By Lemma 4.3 words u and v are conjugated in $F_n(t)$ if and only if $\overline{u} = \overline{v}$. □

We remark that if t is of finite order in $F_n(t)$, then this group is almost free, so, it is word hyperbolic. This implies the solvability of the conjugacy problem in this group. But our approach gives the more effective solving algorithm than the general solving algorithm for word hyperbolic groups.

The following question naturally arises in the context of the above obtained result.

Problem: *Let $G = F \leftthreetimes M$, where $M \subseteq \mathrm{Aut}(F_n)$ is such that the image of M in the group $\mathrm{Out}(F_n) = \mathrm{Aut}(F_n)/\mathrm{Inn}(F_n)$ under the natural homomorphism is finite group. Does the conjugacy problem solvable in G?*

Let us show that the answer on this question is affirmative if M is finite group.

Proposition 4.1. *Let $G = F_n \leftthreetimes M$, where M is a finite subgroup of* Aut (F_n). *Then the conjugacy problem is solvable for G.*

Proof. Let us suppose that M has k elements:

$$M = \{\alpha_0 = e, \alpha_1, \ldots, \alpha_k\}.$$

We order these elements in the following way:

$$\alpha_0 < \alpha_1 < \ldots < \alpha_k.$$

Consider elements $v = \alpha_i V$ and $u = \alpha_j U$ from G, where V and U are reduced words from F_n. Obviously, if elements α_i and α_j are not conjugated in M, then elements v and u are not conjugated in G. So, we can suppose that α_i and α_j are conjugated in M. Using, if necessary, a conjugation, we can assume that $v = \varphi V$ and $u = \varphi U$, where $\varphi \in M$, and φ is taken the smallest representative of the class of conjugated elements that contains α_i and α_j.

Further, as in the proof of Theorem 3.1, we construct sets $D_0(v)$ and $D_0(u)$. In addition, $\Delta \equiv 1$. Define

$$D(v) = \bigcup_{i=0}^{k} D_0(v^{\alpha_i}), \qquad D(u) = \bigcup_{i=0}^{k} D_0(u^{\alpha_i}).$$

For each of these sets we choose a word that has the smallest $\mathbb{X}$-part in respect to the above defined ordering. Such a word will be the conjugated normal form of the corresponding word. Therefore, if obtained words coincide, then elements u and v are conjugated in G. If they are different, then elements u and v are not conjugated in G. □

5. Countably generated free group and shifting automorphism

Let $F_\infty = \langle \ldots, x_{-1}, x_0, x_1, x_2, \ldots \rangle$ be the free group with countably infinite number of generators with words from the alphabet $\mathbb{X} = \{\ldots, x_{-1}^{\pm 1}, x_0^{\pm 1}, x_1^{\pm 1}, x_2^{\pm 1}, \ldots\}$. Consider an automorphism $\varphi : F_\infty \to F_\infty$ acting by shifting generators: $\varphi(x_i) = x_{i-1}$, where $i \in \mathbb{Z}$. We will call φ the *shifting automorphism*. Denote by $F_\infty(\varphi) = F_\infty \ltimes \langle t \rangle$ the split extension, where t is the generator of the cyclic group $\langle t \rangle$ such that the conjugation by t induces the automorphism φ, i.e. $t^{-1}x_i t = x_{i-1}$ for $i \in \mathbb{Z}$. Remark that $F_\infty \triangleleft F_\infty(\varphi)$.

Defining relations for $F_\infty(\varphi)$ can be written as following:

$$\begin{array}{ll} x_i t = t x_{i-1}, & x_i t^{-1} = t^{-1} x_{i+1} \\ x_i^{-1} t = t x_{i-1}^{-1}, & x_i^{-1} t^{-1} = t^{-1} x_{i+1}^{-1}, \end{array}$$

where $i \in \mathbb{Z}$. Using the same arguments as in the proof of Lemma 2.1 from [11], we remark that these relations together with trivial relations form the Gröbner–Shirshov basis for $F_\infty(\varphi)$.

In the present section we will show that the following property holds.

Theorem 5.1. *If ϕ is the shift automorphism of the free group F_∞ then the conjugacy problem is solvable for $F_\infty(\varphi)$.*

Any element of $F_\infty(\varphi)$ is uniquely presented in the form $t^m V$, where $m \in \mathbb{Z}$ and $V \in F_\infty$ is a reduced word. To prove the decidability of the conjugacy problem for the group $F_\infty(\varphi)$ we will show that for each class of conjugated elements we can choose the unique representative (the conjugacy normal form), and that two elements from $F_\infty(\varphi)$ are conjugated if and only if the representatives of corresponding classes coincide.

For any integer m let φ^m be an automorphism of the group F_∞ defined by

$$\varphi^m(x_i) = t^{-m} x_i t^m = x_{i-m}.$$

By $V_{[m]}$ we denote a free reduction of the word $\varphi^m(V)$. A word W is said to be φ^m-*conjugated* to a word V by a word U^{-1} if

$$W \equiv \varphi^m(U) V U^{-1} = U_{[m]} V U^{-1}.$$

Obviously, two words $t^m V$ and $t^m W$ from $F_\infty(\varphi)$ are conjugated by an element of F_∞ if and only if corresponding words V and W from F_∞ are

φ^m–conjugated. A reduced word V is said to be *cyclically φ^m-reduced* if it can not be presented in the form

$$V \equiv \psi^m(U)\, V_0\, U^{-1},$$

for non-trivial $U \in F_\infty$.

Lemma 5.1. *Each element $v \in F_\infty(\varphi)$, presented by a word $t^m V$, is conjugated to a word $t^m V_0$, where V_0 is cyclically φ^m-reduced.*

Proof. If V is not cyclically φ^m-reduced, then v can be presented in the form

$$v \equiv t^m\, \varphi^m(U)\, V_0\, U^{-1},$$

where V_0 is cyclically φ^m-reduced Conjugating this word by U we get

$$U^{-1} v U \equiv U^{-1}\,(t^m \varphi^m(U) V_0 U^{-1})\, U \;=\; U^{-1} t^m t^{-m} U t^m V_0 \;=\; t^m V_0.$$

□

A word $v \equiv t^m\, V \in F_\infty(\varphi)$ is said to be *cyclically φ^m-reduced* if the word $V \in F_\infty$ is cyclically φ^m-reduced.

Suppose that a word $V \in F_\infty$ has some x_k as the final letter, i.e. $V \equiv U x_k$. Let us define the *cyclic φ^m-shift of the final letter*, denoted by τ_m, as following:

$$\tau_m \,:\, V \;\equiv\; U x_k \;\mapsto\; \varphi^m(x_k)\, V\, x_k^{-1} \;\equiv\; \varphi^m(x_k)\,(U x_k)\, x_k^{-1} \;=\; x_{k-m} U.$$

Obviously, if $v = t^m V$ then $v' = t^m \tau_m(V)$ is conjugated to v by x_j^{-1}:

$$x_j v x_j^{-1} \;=\; x_j (t^m U x_j) x_j^{-1} \;=\; t^m x_j t^{-m} t^m U \;=\; t^m x_{j-m} U \;=\; v'.$$

Let us define a linear order "$<$" on the set of reduced words of the alphabet $\mathbb{X}$. Assume that elements of $\mathbb{X}$ are ordered in the following way:

$$\ldots < x_{-1} < x_{-1}^{-1} < x_0 < x_0^{-1} < x_1 < x_1^{-1} < x_2 < x_2^{-1} < \ldots$$

We write $U < V$ if $|U| < |V|$ or if $|U| = |V|$ and the word U is less than the word V in respect to the lexicographical order corresponding to the above defined linear order on $\mathbb{X}$.

Remark the following obvious property:

Lemma 5.2. *Let words $U_1, U_2, \ldots, U_p$ representing elements of F_∞ be ordered in the following way:*

$$U_1 \,<\, U_2 \,<\, \ldots \,<\, U_p,$$

and m be an integer. Then the following ordering holds

$$\varphi^m(U_1) \,<\, \varphi^m(U_2) \,<\, \ldots \,<\, \varphi^m(U_p),$$

that can be also written as

$$U_{1\,[m]} \,<\, U_{2\,[m]} \,<\, \ldots \,<\, U_{p\,[m]}.$$

For a φ^m-reduced word $v \equiv t^m V \in F_\infty(\varphi)$, with $|V| = n$, define a set

$$D(v) = \{t^m \, \tau_m^i(V) \quad | \quad i = 0, 1, \ldots, n-1\}.$$

Let us choose a word from $D(v)$, say $t^m V_0$, such that its $\mathbb{X}$-part V_0 is smallest in respect to above defined lexicographical order. Suppose that V_0 starts from a letter x_k^ε, $\varepsilon = \pm 1$. Then the word $\varphi^k(V_0)$ starts from the letter x_0^ε. For such chosen V_0 and k, the word $\overline{v} = t^m \varphi^k(V_0)$ will be referred to as the *conjugacy normal form* for v. Since the automorphism φ^k acts by the conjugation and the φ^m–conjugation can be realized by the conjugation too, any word is conjugated to its conjugacy normal form.

Lemma 5.3. *If words $v, w \in F_\infty(\varphi)$ are conjugated, then their conjugacy normal forms coincide.*

Proof. Suppose that we have found the conjugacy normal form for a given element $v = t^k V$. By Lemma 5.1, we can assume, up to a conjugation, that v is taken to be cyclically φ^k-reduced. Let us consider an element conjugated to v and demonstrate that their conjugacy normal forms coincide. Indeed, let $u = t^\ell U$ and

$$w = u^{-1} v u \equiv U^{-1} t^{-\ell} t^k V t^\ell U = t^k \, U_{[k]}^{-1} \, V_{[\ell]} U.$$

Consider possible cases of cancellations in the word $U_{[k]}^{-1} \, V_{[\ell]} \, U$.

<u>Case 1.</u> Suppose that there are no cancellations. Then

$$U \, w \, U^{-1} \; = \; U \, t^k \, U_{[k]}^{-1} \, V_{[\ell]} \; = \; t^k \, V_{[\ell]}.$$

Thus, for any m, the cyclic φ^m-shifts of UwU^{-1} can be obtained from the cyclic φ^m-shift of v using conjugation by $t^{-\ell}$. Therefore, a conjugate normal form of w coincides with a conjugate normal form of v.

Let us assume that below $w \equiv t^k W$ is cyclically φ^k-reduced. Using induction by length $|U|$ of U we will show that W can be obtained from V by cyclic φ^k–shift and conjugation by some degree of t. Then there must be some cancellations in the word $U_{[k]}^{-1} \, V_{[\ell]} \, U$.

<u>Case 2.</u> Suppose that there are cancellations in the product $V_{[\ell]} \, U$, i.e. $V_{[\ell]} \equiv V'_{[\ell]} \, V''_{[\ell]}$ and $U \equiv (V''_{[\ell]})^{-1} \, U'$. Then

$$U_{[k]}^{-1} \, V_{[\ell]} \, U \; \equiv \; (U')_{[k]}^{-1} \, V''_{[\ell+k]} \, (V'_{[\ell]} \, V''_{[\ell]} \,)(V''_{[\ell]})^{-1} \, U' \; = \; (U'_{[k]})^{-1} \, V''_{[\ell+k]} \, V'_{[\ell]} \, U'.$$

Remark that there can not be cancellations in the product $V''_{[\ell+k]} \, V'_{[\ell]}$ because, by the assumption, $v = t^k V$ is cyclically φ^k-reduced. Remark that the element $V''_{[\ell+k]} \, V'_{[\ell]}$ can be obtained from v by a cyclic φ^k–shift and a conjugation by t^ℓ. Since $|U'| < |U|$, the induction assumption can be applied. Therefore, its normal form coincides with a conjugated normal form of v.

<u>Case 3.</u> Suppose that there are cancellations in the product $U_{[k]}^{-1} \, V_{[\ell]}$, i.e.

$$V_{[\ell]} \; \equiv \; V'_{[\ell]} \, V''_{[\ell]}, \qquad U_{[k]}^{-1} \; = \; (U')^{-1} \, (V'_{[\ell]})^{-1}, \quad U_{[k]} \; = \; V'_{[\ell]} \, U'.$$

Then

$$U_{[k]}^{-1} V_{[\ell]} U \equiv (U')^{-1} (V'_{[\ell]})^{-1} (V'_{[\ell]} V''_{[\ell]}) V'_{[\ell-k]} U'_{[-k]} = (U')^{-1} V''_{[\ell]} V'_{[\ell-k]} U'_{[-k]},$$

and there are no cancellations in the product $V_{[\ell]} U$. We see that the element $V''_{[\ell]} V'_{[\ell-k]}$ can be obtained from $V' V''$ by a cyclic k–shift, i.e.

$$V'' v (V'')^{-1} = t^k V''_{[k]} V',$$

and a conjugation by $t^{-\ell+k}$. Since $|U'| < |U|$, the induction assumption can be applied, and we get the statement. □

As a consequence of the above considerations we get the statement of Theorem 5.1.

References

[1] S.I. Adyan, V.G. Durnev, *Decision problems for groups and semigroups,* Uspekhi Mat. Nauk **55** (2000), no. 2(332), 3–94 (Russian); translated in Russ. Math. Surveys **55** (2000), no. 2, 207–296.

[2] K. Appel, *On Artin groups and Coxeter groups of large type,* in: Contributions in group theory, 50–78, Contemp. Math. **33**, Amer. Math. Soc., Providence, RI, 1984.

[3] V. Bardakov, L. Bokut, A. Vesnin, *Twisted conjugacy in free groups and Makanin's question,* Seoul National University, 2003, RIM-GARC Preprint Series 03–12, 19 pp., aviliable from arXiv:math.GR/0401349.

[4] K. Bencsath, B. Fine, A.M. Gaglione, A. Myasnikov, F. Roehl, G. Rosenberger, D. Spellman, *Aspects of the theory of free groups,* in: Algorithmic problems in groups and semigroups (Lincoln, NE, 1998), 51–90, Trends Math., Birkhaüser Boston, Boston, MA, 2000.

[5] M. Bestvina, M. Feighn, *A combination theorem for negatively curved groups,* J. Diff. Geom., **35** (1992), no. 1, 85–101; ibid. **43** (1996), no. 4, 783–788.

[6] V.N. Bezverkhnii, *Solution of the conjugacy problem for words in some classes of groups,* Algorithmic problems in the theory of groups and semigroups, Tulsk. Gos. Ped. Inst., Tula, 1990, 103–152 (Russian).

[7] V.N. Bezverkhnii, *Solution of the conjugacy problem for words in Artin and Coxeter groups of large type,* Algorithmic problems in the theory of groups and semigroups, Tulsk. Gos. Ped. Inst., Tula, 1986, 21–61 (Russian).

[8] V.N. Bezverkhnii, I.V. Dobrynina, *Solution of the conjugacy problem for words in Coxeter groups of large type*, Chebyshev Sbornik, **4(1)** (2003), 10–33 (Russian).

[9] L.A. Bokut, *Degrees of unsolvability of the conjugacy problem for finitely presented groups,* Algebra i Logika **7** (1968), no. 5, 4–70; ibid **7** (1968), no. 6, 4–52. (Russian)

[10] L.A. Bokut, G.P. Kukin, *Algorithmic and combinatorial algebra,* Mathematics and its Applications, **255**. Kluwer Academic Publishers Group, Dordrecht, 1994.

[11] L. Bokut, A. Vesnin, *Gröbner–Shirshov bases for some braid groups,* 17 pp. to appear in Journal of Symbolic Computations, (2004).

[12] M.R. Bridson, *The conjugacy and isomorphism problems for combable groups,* Math. Ann. **327** (2003), no. 2, 305–314.

[13] M.R. Bridson, A. Haefliger, *Metric spaces of non-positive curvature,* Grundl. Math. Wiss., **319**, Springer-Verlag, Berlin–Heidelberg, 1999.

[14] E. Brieskorn, K. Saito, *Artin-Gruppen und Coxeter-Gruppen,* Invent. Math. **17** (1972), 245–271.

[15] P. Brinkmann, *Hyperbolic automorphisms of free groups,* Geom. Funct. Anal. **10** (2000), no. 5, 1071–1089.

[16] D. Collins, C. Miller III, *The conjugasy problem and subgroups of finite index,* Proc. London Math. Soc. (3) **34** (1977), no. 3, 535–556.

[17] M. Dehn, *Über unendichle diskontinuerliche Gruppen,* Math. Ann. **71** (1912), 116–144.

[18] J.L. Dyer, *Separating conjugates in free-by-finite groups,* J. London Math. Soc. (2) **20** (1979), no. 2, 215–221.

[19] J.L. Dyer, *Separating conjugates in amalgamated free products and HNN extensions,* J. Austral. Math. Soc., Ser A **29** (1980), no. 1, 35–51.

[20] M. Feighn, M. Handel, *Mapping tori of free group automorphisms are coherent,* Ann. of Math. (2) **149** (1999), no. 3, 1061–1077.

[21] A.A. Fridman, *A solution of the conjugacy problem in a certain class of groups,* in: Mathematical logic, theory of algorithms and theory of sets (dedicated to P. S. Novikov on the occasion of his seventieth birthday). Proc. Steklov Inst. Math. **133** (1977), 233–242, 276.

[22] F.A. Garside, *The braid group and other groups,* Quart. J. Math. Oxford Ser. (2) **20** (1969), 235–254.

[23] S.M. Gersten, J.R. Stallings, *Irreducible outer automorphisms of a free group,* Proc. Amer. Math. Soc., **111** (1991), no. 2, 309–314.

[24] A.V. Goryaga, A.S. Kirkinskii, *The decidability of the conjugacy problem cannot be transferred to finite extensions of groups,* Algebra i Logika **14(4)** (1975), 393–406. (Russian)

[25] M. Gromov, *Hyperblic groups,* in: Essays in Group Theory, Ed. S.M. Gersten, Math. Sci. Res. Inst. Publ. **8**, Springer, New-York, 1987, 75–263.

[26] G.A. Gurevich, *On the conjugacy problem for groups with one defining relation,* Sov. Math., Dokl. **13** (1972), 1436–1439.

[27] G.A. Gurevich, *On the conjugacy problem for groups with a single defining relation,* in: Mathematical logic, theory of algorithms and theory of sets (dedicated to P. S. Novikov on the occasion of his seventieth birthday). Proc. Steklov Inst. Math. **133** (1977), 108–120.

[28] R.D. Hurwitz, *A survey of the conjugacy problem,* Contributions to group theory, 278–298, Contemp. Math., **33**, Amer. Math. Soc., Providence, RI, 1984.

[29] I. Kapovich, *Mapping tori of endomorphisms of free groups,* Comm. in Algebra **28** (2000), no. 6, 2895–2917.

[30] O.G. Kharlampovich, M.V. Sapir, *Algoritmic problems in varieties,* Internat. J. Algebra Comput. **5** (1995), no. 4-5, 379–602.

[31] *The Kourovka notebook. Unsolved problems in group theory.* Fifteenth augmented edition. Edited by V.D. Mazurov and E.I. Khukhro. Russian Academy of Sciences Siberian Division, Institute of Mathematics, Novosibirsk, 2002. 125 pp.

[32] L. Larsen, *The conjugacy problem and cyclic HNN constructions,* J. Austral. Math. Soc. Ser. A **23** (1977), no. 4, 385–401.

[33] R.C. Lyndon, P.E. Schupp, *Combinatorial group theory.* Ergebnisse der Mathematik und ihrer Grenzgebiete, **89**. Springer-Verlag, Berlin-New York, 1977.

[34] A.I. Malcev, *On homomorphisms onto finite groups,* Uchen. Zapiski Ivanovsk. ped. instituta **18** (1958), no. 5, 49–60 (also in "Selected papers", Vol. 1, Algebra, (1976), 450–461)

[35] A.I. Malcev, *On isomorphi matrix representations of infinite groups,* Mat. Sbornik **8** (1940), no. 3, 405–422.

[36] B. Newman, *Some results on one-relator groups,* Bull. Amer. Math. Soc. **74** (1968), 568–571.

[37] G.A. Noskov, V.N. Remeslennikov, V.A. Roman'kov, *Infinite groups,* Itogi Nauki Tekh., Ser. Algebra Topologiya Geom. **17** (1979), 65–157 (Russian).

[38] P.S. Novikov, *Unsolvability of the conjugacy problem in the theory of groups,* Izv. Akad. Nauk SSSR. Ser. Mat. **18** (1954), 485–524 (Russian).

[39] S. Pride, *Small cancellation conditions satisfied by one-relator groups,* Math. Z. **184** (1983), no. 2, 283–286.
[40] Z. Sela, *The conjugcy problem for knot groups,* Topology **32** (1993), no. 2, 363–369.
[41] C.M. Weinbaum, *The word and conjugacy problems for the knot group of any tame, prime, alternating knot,* Proc. Amer. Math. Soc. **30** (1971), 22–26.

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